Appendix

We are working in the sagittal (z, y) plane and assume that the section of the eye has the form of an ellipse with the semiaxes a and b,

$$\frac{z^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{1}$$

The fovea has the coordinates $(-a, 0)$. Consider the upper half of the ellipse.

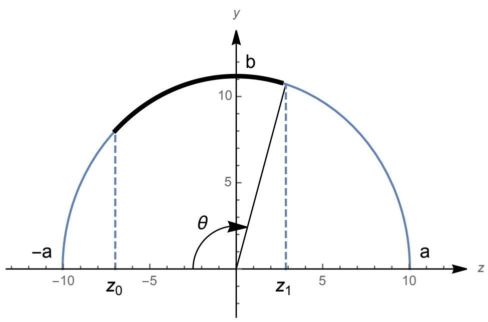

Figure 1: Stage

In polar coordinates, the upper-half of the ellipse can be parametrized as follows

$$z = -R(\theta)\,\cos\theta\,, \quad y = R(\theta)\,\sin\theta\,, \quad 0 \le \theta \le \pi\,,$$

where

$$R(\theta) = \frac{ab}{\sqrt{b^2\,\cos^2\theta + a^2\,\sin^2\theta}} \tag{2}$$

However, this parametrization is inconvenient for our calculations. Instead, we view the whole arc as the graph of the function

$$y = f(z) = b\sqrt{1 - \frac{z^2}{a^2}}\,, \tag{3}$$

where z runs from $-a$ to a. For any value z_1 between $-a$ and a, the length of the arc between the points $(-a, 0)$ and (z_1, y_1), where $y_1 = f(z_1)$, is given by

$$L_e(z_1) = \int_{-a}^{z_1} \sqrt{1 + \left(\frac{df(z)}{dz}\right)^2}\, dz \underset{z=-a\,\cos\,t}{=} b \int_0^{\arccos(-z_1/a)} \sqrt{1 - (1 - \frac{a^2}{b^2})\,\sin^2 t}\, dt\,.$$

Denoting by $E(\phi, m)$ the elliptic integral of the second kind, $E(\phi, m) = \int_0^\phi \sqrt{1 - m\, \sin^2 t}\, dt$, we see that

$$L_e(z_1) = b\, E\left(\arccos(-\frac{z_1}{a}),\, 1 - \frac{a^2}{b^2}\right)\,. \tag{4}$$

<u>**Claim:**</u> For any two points z_0 and z_1, $-a \le z_0 < z_1 \le a$, there exists a unique (convex) catenary tethered at the points (z_0, y_0) and (z_1, y_1) on the (upper half of the) ellipse and such that its length between z_0 and z_1 is equal to the length of the arc of the ellipse between the same points.

Recall that the equation of the general convex (hanging down) catenary is

$$y(z) = -\lambda + s \cdot \cosh(\frac{z + \alpha}{s}) \tag{5}$$

There are three parameters: $s > 0$, α, and λ, and we claim that they can be chosen in a unique way for the catenary to satisfy our requirements, which are

1. $y(z)$ connects two points, (z_0, y_0) and (z_1, y_1), on the ellipse.
2. The length of the catenary (5) between z_0 and z_1 is equal to the length of the arc between (z_0, y_0) and (z_1, y_1) on the ellipse.

The first requirement means

$$-\lambda + s\, \cosh(\frac{z_0 + \alpha}{s}) = y_0 = b\sqrt{1 - \frac{z_0^2}{a^2}}\,, \tag{6}$$

and

$$-\lambda + s\, \cosh(\frac{z_1 + \alpha}{s}) = y_1 = b\sqrt{1 - \frac{z_1^2}{a^2}}\,. \tag{7}$$

The second requirement is

$$L_e(z_1) - L_e(z_0) = L_{cat}(z_0, z_1)\,, \tag{8}$$

where $L_{cat}(z_0, z_1)$ is the arc length of the catenary (5) between z_0 and z_1. An easy to calculation shows that

$$L_{cat}(z_0, z_1) = s\, \left[\sinh(\frac{z_1 + \alpha}{s}) - \sinh(\frac{z_0 + \alpha}{s})\right]\,. \tag{9}$$

Denote

$$\Delta y = y_1 - y_0\,, \quad \Delta z = z_1 - z_0\,, \quad \Delta L = L_e(z_1) - L_e(z_0)\,. \tag{10}$$

It follows from (6) and (7) that

$$\cosh(\frac{z_1 + \alpha}{s}) - \cosh(\frac{z_0 + \alpha}{s}) = \frac{\Delta y}{s}\,. \tag{11}$$

And it follows from (8), that

$$\sinh(\frac{z_1+\alpha}{s}) - \sinh(\frac{z_0+\alpha}{s}) = \frac{\Delta L}{s}\,. \tag{12}$$

Note that

$$\Delta L \geq \Delta y\,.$$

Then equations (11) and (12) can be solved with respect to $\frac{z_1+\alpha}{s}$ and $\frac{z_0+\alpha}{s}$ in terms of Δy, ΔL, and s:

$$\frac{z_1+\alpha}{s} = \ln\left[\frac{1}{2s}\left\{(\Delta L+\Delta y) + \sqrt{(\Delta L)^2-(\Delta y)^2}\,\frac{\sqrt{(\Delta L)^2-(\Delta y)^2+4s^2}}{\Delta L-\Delta y}\right\}\right] \tag{13}$$

and

$$\frac{z_0+\alpha}{s} = \ln\left[\frac{1}{2s}\left\{-(\Delta L+\Delta y) + \sqrt{(\Delta L)^2-(\Delta y)^2}\,\frac{\sqrt{(\Delta L)^2-(\Delta y)^2+4s^2}}{\Delta L-\Delta y}\right\}\right] \tag{14}$$

Subtract (14) from (13) and multiply the result by s:

$$\Delta z = s\cdot\ln\left[1+\frac{1}{2s^2}\left((\Delta L)^2-(\Delta y)^2+\sqrt{(\Delta L)^2-(\Delta y)^2}\,\sqrt{(\Delta L)^2-(\Delta y)^2+4s^2}\right)\right] \tag{15}$$

We use this equation to solve for s. There is a unique solution because. for any $A>0$, the function

$$g(s) = s\,\ln[1+\frac{1}{2s^2}\,(A+\sqrt{A}\,\sqrt{A+4s^2})]\,,\quad s>0\,, \tag{16}$$

is concave and monotone increasing from 0 at $s=0$ to its asymptotic value $\sqrt{A}$ as s goes to infinity. Note that $A=(\Delta L)^2-(\Delta y)^2$ in our case and that $\Delta z<\sqrt{A}$ because the arc of the chord connecting two points on the ellipse is shorter than the corresponding arc of the ellipse, $(\Delta z)^2+(\Delta y)^2<(\Delta L)^2$.

After solving equation (15) for s, we turn to equation (13) and solve for α:

$$\alpha = -z_1 + s\cdot\ln\left[\frac{1}{2s}\left\{(\Delta L+\Delta y) + \sqrt{(\Delta L)^2-(\Delta y)^2}\,\frac{\sqrt{(\Delta L)^2-(\Delta y)^2+4s^2}}{\Delta L-\Delta y}\right\}\right] \tag{17}$$

Next, from equation (7) find λ:

$$\lambda = s\,\cosh(\frac{z_1+\alpha}{s}) - y_1\,. \tag{18}$$

In the case $z_0=-a$, it may be easier to get α from (14),

$$\alpha = a + s\cdot\ln\left[\frac{1}{2s}\left\{-(\Delta L+\Delta y) + \sqrt{(\Delta L)^2-(\Delta y)^2}\,\frac{\sqrt{(\Delta L)^2-(\Delta y)^2+4s^2}}{\Delta L-\Delta y}\right\}\right] \tag{19}$$

and then find λ from (6):

$$\lambda = s\,\cosh(\frac{-a+\alpha}{s})\,. \tag{20}$$

Apart from graphing the curves, there are two places where a somewhat sophisticated computational software is needed: one is to evaluate the elliptic integrals $L_e(z)$, and the other one is to solve equation (15). We used Mathematica for all numerical tasks.

Within the same framework, the second question we dealt with was as follows. Given the right tether point (z_1, y_1) on the ellipse, find the left tether point (z_0, y_0) such that the corresponding catenary just touches the vision axis z.

First, we notice that not for all z_1 there is a solution. Fix the left end of the catenary at the point $(-a, 0)$ (this means $z_0 = -a$ in the formulas above). If z_1 is very close to the left-most position $-a$, the arc of ellipse from $(-a, 0)$ to (z_1, y_1) is too short for the catenary to hang below the z-axis. One can show that the slope of the catenary at $z = -a$ will be strictly positive when z_1 is close to $-a$. On the other hand, when $z_1 = a$, the whole catenary will hang below the z-axis, and the slope of the catenary at $z = -a$ is strictly negative in this case. Because the slope at $z = -a$ changes continuously (monotone decreasing) as we move z_1 from $-a$ to a, there must be a value z_* such that the catenary tethered at the points $(-a, 0)$ and (z_*, y_*) on the ellipse, has the slope $= 0$ at $z = -a$. For the catenary (5) the slope at z is

$$\frac{dy(z)}{dz} = \sinh(\frac{z+\alpha}{s})\,.$$

It is zero when $z + \alpha = 0$. For this to happen when $z = -a$ we must have $\alpha = a$. Then $\lambda = s$ as follows from (20). And there are a few other equalities that must hold. But this does not help in finding z_* analytically, it must be found numerically. We have done this for specific values of a and b.

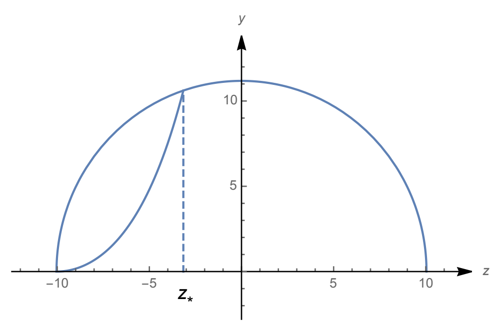

Figure 2: Touch

Thus, for all $z_1 \leq z_*$ and $-a \leq z_0 < z_1$, the corresponding catenary cannot hang below the z-axis.

The continuity argument again shows that when $z_1 > z_*$, there is a critical z_1^* in the interval $(-a, z_1)$ such that for all z_0 in $(z_1^*, z_1]$ the catenary is too short even to reach the

z-axis. For $z_0 = z_1^*$, the minimum of the catenary will be zero, $y_{min} = 0$, the catenary will touch the z-axis. If $z_0 < z_1^*$, the minimum will be negative, $y_{min} < 0$. Note that in this case

$$y_{min} = -\lambda + s \,. \tag{21}$$

We should mention that the value of z_1^* depends on z_1, not only on a and b. Finding z_1^* (or the corresponding y_1^* value) can be done numerically.

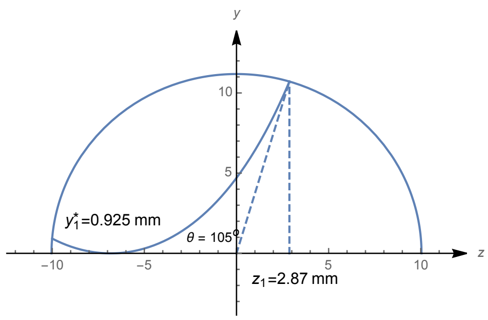

Figure 3: Touch 2